

GROWTH OF BALLS OF HOLOMORPHIC SECTIONS ON PROJECTIVE TORIC VARIETIES

Mounir Hajli *

Abstract

Let $\mathcal{O}(D)$ be an equivariant line bundle which is big and nef on a complex projective nonsingular toric variety X . Given a continuous toric metric $\|\cdot\|$ on $\mathcal{O}(D)$, we define the energy at equilibrium of $(X, \phi_{\overline{D}})$ where $\phi_{\overline{D}}$ is the weight of the metrized toric divisor $\overline{D} = (D, \|\cdot\|)$. We show that this energy describes the asymptotic behaviour as $k \rightarrow \infty$ of the volume of the L^2 -norm unit ball induced by $(X, k\phi_{\overline{D}})$ on the space of global holomorphic sections $H^0(X, \mathcal{O}(kD))$.

Key Words: Toric varieties, Equilibrium weight, Energy functional, Bernstein-Markov property, Monge-Ampère operator.

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1 INTRODUCTION

Let Q be a free \mathbb{Z} -module of rank n and P its dual. We consider a fan Σ on $Q_{\mathbb{R}} = Q \otimes_{\mathbb{Z}} \mathbb{R}$ and we denote by $X = X_{\Sigma}$ the associated toric variety over \mathbb{C} , see for instance [8]. In the sequel, we assume that X is nonsingular and projective (this is equivalent to the fact that Σ is nonsingular and the support of Σ is $Q_{\mathbb{R}}$, see [8, theorems 1.10, 1.11]). We set $\mathbb{T}_Q := \text{Hom}_{\mathbb{Z}}(P, \mathbb{C}^*) \simeq (\mathbb{C}^*)^n$ and we denote by $\mathbb{S}_Q \simeq (\mathbb{S}^1)^n$ its compact torus. We have an open dense immersion $\mathbb{T}_Q \hookrightarrow X$ with an action of \mathbb{T}_Q on X which extends the action of \mathbb{T}_Q on its self by translations.

The toric varieties have a rich geometry that can be related to the geometry of polytopes. Many results in algebraic geometry and complex differential geometry can be tested on them, for instance the Riemann-Roch formula.

On toric varieties, some properties of line bundles can be interpreted in terms of convex geometry. Let D be an equivariant Cartier divisor on X also called a toric divisor, that is a Cartier divisor which is invariant under the action of the torus \mathbb{T}_Q . Let s_D be the rational section of $\mathcal{O}(D)$ associated to D . We know that D defines a Σ -linear support function Ψ_D on Σ (see [8, Definition p. 66]) and D is uniquely defined by this function (see [8, Proposition 2.1 (v)]). Moreover the function Ψ_D defines a convex polytope:

$$\Delta_D := \{x \in P_{\mathbb{R}} \mid \langle x, u \rangle \geq \Psi_D(u), \forall u \in Q_{\mathbb{R}}\},$$

where $P_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R}$. Δ_D and ψ_D encode many geometric informations about D , for instance

1. $H^0(X, \mathcal{O}(D)) = \oplus_{\mathbf{e} \in \Delta_D \cap P} \mathbb{C} \chi^{\mathbf{e}}$, where $\chi^{\mathbf{e}}$ denotes the character associated to \mathbf{e} , see [8, Lemma 2.3].

*Institute of Mathematics, Academia Sinica, Taipei 106, Taiwan *E-mail:* hajli@math.sinica.edu.tw, hajlimounir@gmail.com

2. $\text{vol}(\mathcal{O}(D)) = n! \text{vol}(\Delta_D)$, (this follows from (1.)).
3. D is nef if and only if Ψ_D is concave (see [8, Theorem 2.7]).
4. If D is nef then $\deg_D(X) = n! \text{vol}(\Delta_D)$.
5. D is big if and only if $\dim(\Delta_D) = n$, by (2.).
6. D is ample if and only if Ψ_D is strictly concave (see [8, Theorem 2.13]).

If we denote by X_{\geq}° the quotient of X by \mathbb{S}_Q . Then the open subset X_{\geq}° can be identified with $\text{Hom}_{\mathbb{Z}}(P, \mathbb{R}_{>0})$ (see [6, §4]). But, since $Q_{\mathbb{R}} = \text{Hom}_{\mathbb{Z}}(P, \mathbb{R})$ then the usual exponential defines a morphism $\exp(-(\cdot)) : Q_{\mathbb{R}} \rightarrow X_{\geq}^{\circ} \hookrightarrow X(\mathbb{C})$. This morphism extends in a obvious way to $Q_{\mathbb{C}} \rightarrow \mathbb{T}_Q \hookrightarrow X(\mathbb{C})$.

We say that a function f on X (resp. a hermitian metric $\|\cdot\|$ on $\mathcal{O}(D)$) is \mathbb{S}_Q -invariant if $f(z \cdot x) = f(x)$ (resp. $\|s_D\|(z \cdot x) = \|s_D\|(x)$) for any $x \in X$ and $z \in \mathbb{S}_Q$. A continuous hermitian metric $\|\cdot\|$ on $\mathcal{O}(D)$ is said semipositive if the Chern current $c_1(\mathcal{O}(D), \|\cdot\|)$ is nonnegative. An admissible metric is by definition a uniform limit of a sequence of smooth hermitian semipositive metrics. When $\mathcal{O}(D)$ is ample then the notions of admissible metrics and semipositives metrics are equivalent, see [7, Theorem 4.6.1].

The function Ψ_D defines a continuous hermitian metric on $\mathcal{O}(D)$ called the *canonical metric* of $\mathcal{O}(D)$. It is given locally as follows: The norm of a local holomorphic section s at a point x is the following real

$$\|s(x)\|_{\infty, D} = \left| \frac{s}{\chi^{\Psi_D}}(x) \right|.$$

(see [7, Proposition 3.3.1]). When $\mathcal{O}(D)$ is spanned by its global sections (equivalently Ψ_D is concave) one can show that $\|\cdot\|_{D, \infty} = \phi_D^* \|\cdot\|_{\infty}$ where $\phi_D : X \rightarrow \mathbb{P}^{\#(\Delta_D \cap P)-1}$ is the equivariant morphism defined in terms of $\Delta_D \cap P$ and $\|\cdot\|_{\infty}$ is the canonical metric of $\mathcal{O}(1)$ (see [7, §3.3.3] for a detailed construction). Moreover we have that $\|\cdot\|_{\infty, D}$ is a uniform limit of a sequence of smooth semipositive metrics and $-\log \|s_D\|_{\infty, D}^2$ is a plurisubharmonic weight on $\mathcal{O}(D)$ [7, Propositions 3.3.11, 3.3.12].

Let $\|\cdot\|_{\overline{D}}$ be a \mathbb{S}_Q -invariant hermitian metric on $\mathcal{O}(D)$ such that $\|\cdot\|_{\overline{D}} / \|\cdot\|_{\infty, D}$ is bounded on X . We let $\overline{D} := (D, \|\cdot\|_{\overline{D}})$ the obtained hermitian line bundle and we called it a *toric metrized divisor*. We set $g_{\overline{D}} : Q_{\mathbb{R}} \rightarrow \mathbb{R}$ the function defined as follows:

$$g_{\overline{D}}(u) := \log \|s_D(\exp(-u))\|_{\overline{D}} \quad \forall u \in Q_{\mathbb{R}}.$$

By definition of the canonical metric one can check easily that

$$g_{\overline{D}_{\infty}}(u) = \inf\{ \langle v, u \rangle \mid v \in \Delta_D \} \quad \forall u \in Q_{\mathbb{R}}. \quad (1)$$

($\langle v, u \rangle$ denotes the pairing defined by $Q_{\mathbb{R}}$ and $P_{\mathbb{R}}$). We denote by $\check{g}_{\overline{D}} : P_{\mathbb{R}} \rightarrow [-\infty, +\infty[$ the Legendre-Fenchel transform of $g_{\overline{D}}$, i.e the function defined for any $x \in P_{\mathbb{R}}$ as follows

$$\check{g}_{\overline{D}}(x) := \inf_{u \in Q_{\mathbb{R}}} (\langle x, u \rangle - g_{\overline{D}}(u)).$$

We have $\check{g}_{\overline{D}_{\infty}}$ vanishes on Δ_D and equal to $-\infty$ otherwise (One can show that this follows from the following assertion: $\check{g}_{\overline{D}_{\infty}}(x) = t \check{g}_{\overline{D}_{\infty}}(x)$ for any $x \in \Delta_D$ and $t > 0$, which is an easy consequence of (1)). Combining this with Proposition 2.3, we can show that $\check{g}_{\overline{D}}(x)$ is finite if and only if $x \in \Delta_D$ and $\check{g}_{\overline{D}}$ is concave on Δ_D .

Let $\|\cdot\|_{\overline{D}_0}$ and $\|\cdot\|_{\overline{D}_1}$ be two smooth hermitian metrics on D , $\phi_{\overline{D}_0}$ and $\phi_{\overline{D}_1}$ the associated weights. We define the Monge-Ampère functional \mathcal{E} by the formula

$$\mathcal{E}(\overline{D}_1) - \mathcal{E}(\overline{D}_0) = \frac{1}{n+1} \sum_{j=0}^n \int_X -\log \frac{\|\cdot\|_{\overline{D}_1}}{\|\cdot\|_{\overline{D}_0}} c_1(\overline{D}_0)^{\wedge j} \wedge c_1(\overline{D}_1)^{\wedge n-j}.$$

By the theory of Bedford-Taylor [1], this definition extends to admissible metrics, and hence to integrable ones by polarisation. By definition an integrable metric can be written, in additive notation, as a difference of two admissible metrics.

Following [2], when $\mathcal{O}(D)$ is big we set

$$\mathcal{E}_{\text{eq}}(\overline{D}_1) - \mathcal{E}_{\text{eq}}(\overline{D}_0) := \frac{1}{\text{Vol}(D)} (\mathcal{E}((\overline{D}_1)_X) - \mathcal{E}((\overline{D}_0)_X)).$$

where $(\overline{D}_i)_X$ is the metrized toric divisor D endowed with the weight $P_X \phi_{\overline{D}_i}$, the equilibrium weight of $\phi_{\overline{D}_i}$ for $i = 0, 1$. In [2, §1.3], $\mathcal{E}_{\text{eq}}(\overline{D})$ is called the energy at equilibrium of $(X, \phi_{\overline{D}})$ ($\phi_{\overline{D}}$ is the weight of \overline{D}).

Our first result is Theorem 3.4 which gives an integral representation of the variation of the energy functional \mathcal{E} in terms of some combinatorial objects defined on the polytope associated to D . This theorem can be seen as a toric version of [2, Theorem B].

Let μ be a probability measure with non-pluripolar support on X . We endow the space of global sections $H^0(X, \mathcal{O}(D))$ with the L^2 -norm

$$\|s\|_{L^2(\mu, \overline{D})}^2 := \int_X \|s\|_{\overline{D}}^2 \mu.$$

Also we consider the sup norm defined as follows

$$\|s\|_{\text{sup}, \overline{D}} := \sup_{x \in X} \|s\|_{\overline{D}}(x).$$

for any $s \in H^0(X, \mathcal{O}(D))$. Let $k \in \mathbb{N}^*$. We consider the following functional

$$\mathcal{L}_k(\mu, k\overline{D}) := \frac{1}{2kN_k} \log \text{vol}_k B^2(\mu, k\overline{D}),$$

where $\text{vol}_k B^2(\mu, k\overline{D})$ is by definition the volume of the unit ball $B^2(\mu, k\overline{D})$ in $H^0(X, \mathcal{O}(kD))$ with respect to the L^2 -norm, and $N_k := \dim H^0(X, \mathcal{O}(kD))$.

The Bergman distortion function $\rho(\mu, \overline{D})$ is by definition the function given at a point $x \in X$ by

$$\rho(\mu, \overline{D})(x) = \sup_{s \in H^0(X, \mathcal{O}(D)) \setminus \{0\}} \frac{\|s(x)\|_{\overline{D}}^2}{\|s\|_{L^2(\mu, \overline{D})}^2}.$$

If $\{s_1, \dots, s_N\}$ is a $L^2(\mu, \overline{D})$ -orthonormal basis of $H^0(X, \mathcal{O}(D))$, then it is well known that

$$\rho(\mu, \overline{D})(x) = \sum_{j=1}^N \|s_j(x)\|_{\overline{D}}^2 \quad \forall x \in X.$$

Definition 1.1. We say that μ has the Bernstein-Markov property with respect to $\|\cdot\|_{\overline{D}}$ if $\forall \varepsilon > 0$ we have

$$\sup_X \rho(\mu, k\overline{D})^{\frac{1}{2}} = O(e^{k\varepsilon}).$$

If μ is a smooth positive volume form and $\|\cdot\|_{\overline{D}}$ is a continuous metric on $\mathcal{O}(D)$ then μ has the Bernstein-Markov property with respect to $\|\cdot\|_{\overline{D}}$ (see [2, Lemma 3.2]).

Our main result is the following theorem:

Theorem 1.2. [Main theorem] Let X be a complex projective nonsingular toric variety and D a toric divisor on X such that $\mathcal{O}(D)$ is big and nef. Let $\overline{D}_i := (D, \|\cdot\|_{\overline{D}_i})$ be a continuous toric metrized divisor on X for $i = 0, 1$. Let μ_j be a probability measure which is \mathbb{S}_Q -invariant on X and with the Bernstein-Markov property with respect to $\|\cdot\|_{\overline{D}_j}$, $j = 0, 1$. Then as $k \rightarrow \infty$ we have

$$\mathcal{L}_k(\mu_1, \overline{D}_1) - \mathcal{L}_k(\mu_0, \overline{D}_0) \rightarrow \mathcal{E}_{\text{eq}}(\overline{D}_1) - \mathcal{E}_{\text{eq}}(\overline{D}_0).$$

This theorem describes the asymptotics of $\mathcal{L}_k(\mu, \overline{D})$, the functional volume of the balls of the holomorphic sections of a continuous toric divisor \overline{D} when k tends to ∞ . In particular we recover partially a result of Berman and Boucksom [2, Theorem A]. Comparing to [2] our approach is completely different. In fact, our strategy is based mainly on the combinatorial structure of the toric variety, which makes the proof much easier. A crucial ingredient in the proof of Theorem 1.2 is Theorem 3.4.

2 The Monge-Ampère operator and the equilibrium weight

We keep the same notations as in the introduction. Let X be a complex projective nonsingular toric variety and L a holomorphic line bundle over X . Let ϕ be a weight of a continuous hermitian metric $e^{-\phi}$ on L . When ϕ is smooth we define the Monge-Ampère operator as

$$\text{MA}(\phi) := (dd^c \phi)^{\wedge n}.$$

The equilibrium weight of ϕ is defined as:

$$P_X \phi := \sup\{\psi \mid \psi \text{ psh weight on } L, \psi \leq \phi \text{ on } X\}.$$

It is known that the equilibrium weight is upper semicontinuous psh weight with minimal singularities.

Proposition 2.1. Let ϕ_1 and ϕ_0 be two continuous weights on L a big line bundle on X . We have

$$|P_X \phi_1 - P_X \phi_0| \leq \sup_{x \in X} |\phi_1 - \phi_0|.$$

Proof. This is an easy consequence of the definition of the equilibrium weight. □

Assume that L is ample and let ω be a positive $(1, 1)$ -form in $c_1(L)$. We set

$$\mathcal{H}_\omega := \{u \in \mathcal{C}^\infty(X) \mid dd^c u + \omega > 0\}.$$

Clearly, \mathcal{H}_ω is a convex subset which is identified with the set of smooth positive hermitian metrics (resp. weights) on L . We set $P_\omega(u) := \sup\{v \in \mathcal{H}_\omega \mid v \leq u\}$. Following [2], [3], this operator extends to $\mathcal{C}^0(X)$ with image in $\mathcal{C}^0(X) \cap \overline{\mathcal{H}_\omega}$. In other words, if ϕ is a continuous weight on L an ample line bundle then $P_X \phi$ is also continuous.

Remark 2.2. Let D be a toric nef divisor on X . Let $\|\cdot\|_{\overline{D}}$ (resp. ϕ_D) a continuous hermitian metric (resp. continuous weight) on D . Then $\|\cdot\|_{\overline{D}_X}/\|\cdot\|_{\infty,D}$ (resp. $P_X\phi_D - \phi_{\infty,D}$) is bounded on X , where \overline{D}_X is hermitian line bundle $\mathcal{O}(D)$ endowed with the metric defined by $P_X\phi_D$. Indeed, since D is nef then $\|\cdot\|_{\infty,D}$ is a semipositive metric. Then we can find a constant C such that $\phi_{\infty,D} - C \leq P_X\phi \leq \phi_D$.

Proposition 2.3. Let g be a real function on $Q_{\mathbb{R}}$. Then g defines a hermitian (continuous) metric $\|\cdot\|_g$ on $\mathcal{O}(D)$ if and only if $g - \Psi_D$ extends to a bounded (continuous) function on X_{\geq} . Moreover, we have

$$\sup_{x \in \Delta_D} |\check{g} - \check{g}'| \leq \sup_{u \in Q_{\mathbb{R}}} |g - g'|,$$

for any g and g' two functions on $Q_{\mathbb{R}}$ defining hermitian metrics on $\mathcal{O}(D)$.

Proof. The proof is an easy consequence of the definitions. \square

Lemma 2.4. Let $\overline{D} = (D, \|\cdot\|_{\overline{D}})$ be a continuous metrized divisor such that $\mathcal{O}(D)$ is big and nef on X . We set $\phi_D := -\log \|s_D\|_{\overline{D}}$ and we denote by \overline{D}_X the metrized toric divisor D endowed with $P_X\phi_D$. Then $P_X\phi_D$ is a \mathbb{S}_Q -invariant weight on D and the following equality holds on Δ_D

$$\check{g}_{\overline{D}} = \check{g}_{\overline{D}_X}.$$

Proof. By definition ϕ_D is \mathbb{S}_Q -invariant weight. Let ψ be a psh weight on $\mathcal{O}(D)$ such that $\psi \leq \phi$. For any $t \in \mathbb{S}_Q$, we set $\psi_t := \psi(t \cdot (\cdot))$. Then ψ_t is a psh weight verifying $\psi_t(z) \leq \phi_D(t \cdot z) = \phi_D(z)$ for any $z \in X$. That is $\psi_t \leq \phi_D$. We conclude that $(P_X\phi_D)_t \leq P_X\phi_D$ for any $t \in \mathbb{S}_Q$. It follows that $P_X\phi_D$ is a \mathbb{S}_Q -invariant weight on D . By (2.2), $\check{g}_{\overline{D}_X}$ is well defined. Moreover, we have $\|s\|_{\sup,k\overline{D}} = \|s\|_{\sup,k\overline{D}_X}$ for any $k \in \mathbb{N}$ and $s \in H^0(X, \mathcal{O}(kD))$ (see for instance [2, Proposition 2.8]). Let $k \in \mathbb{N}^*$ and $\mathbf{e} \in k\Delta_D \cap P$. We have $\|\chi^{\mathbf{e}}\|_{\sup,k\overline{D}} = \sup_{x \in X} \|\chi^{\mathbf{e}}(x)\|_{k\overline{D}} = \sup_{u \in Q_{\mathbb{R}}} \|\chi^{\mathbf{e}}(\exp(-u))\|_{k\overline{D}} = \exp(-k \inf_{u \in Q_{\mathbb{R}}} (\frac{\mathbf{e}}{k} \cdot u - g_{\overline{D}}(u))) = \exp(-k\check{g}_{\overline{D}}(\frac{\mathbf{e}}{k}))$. We deduce that

$$\check{g}_{\overline{D}}(x) = \check{g}_{\overline{D}_X}(x) \quad \forall x \in \Delta_D \cap \mathbb{Q}^n.$$

Using the fact that a concave and finite function on a Δ_D is necessarily continuous on its interior, see [9, Theorem 10.1], then we get $\check{g}_{\overline{D}} = \check{g}_{\overline{D}_X}$ on $\text{Int}(\Delta_D)$. But, since Δ_D is the convex closure of $\Delta_D \cap P$ then

$$\check{g}_{\overline{D}}(x) = \check{g}_{\overline{D}_X}(x) \quad \forall x \in \Delta_D.$$

\square

3 The energy functional in the toric setting

The goal of this section is to give a formula for the variation of the energy functional \mathcal{E} in terms of the Legendre-Fenchel transform. First, this formula is proved in the ample case, see Theorem 3.4, then we deduce the general case of big and nef divisors in Corollary 3.6.

Proposition 3.1. Let D be a toric divisor on X . Assume that there exists $\|\cdot\|$ an admissible and \mathbb{S}_Q -invariant metric on $\mathcal{O}(D)$. Then there exists a sequence of smooth, semipositive and \mathbb{S}_Q -invariant hermitian metrics converging uniformly to $\|\cdot\|$.

Proof. First let recall that given a smooth hermitian metric $\|\cdot\|$ one can average it in order to get a \mathbb{S}_Q -invariant smooth metric. This is done as follows, we define the metric $\|\cdot\|_{\mathbb{S}_Q}$ given on \mathbb{T}_Q by $\log \|s(x)\|_{\mathbb{S}_Q} = \int_{\mathbb{S}_Q} \log \|s(t \cdot x)\| d\mu_{\text{Haar}}$. This metric extends to X since $\log(\|s(x)\|_{\mathbb{S}_Q}/\|s(x)\|') = \int_{\mathbb{S}_Q} \log(\|s(t \cdot x)\|/\|s(x)\|') d\mu_{\text{Haar}}$ where $\|\cdot\|'$ is a smooth and \mathbb{S}_Q -invariant metric, extends to a smooth

function to X . Clearly the metric $\|\cdot\|_{\mathbb{S}_Q}$ is \mathbb{S}_Q -invariant and smooth. Moreover $c_1(\mathcal{O}(D), \|\cdot\|_{\mathbb{S}_Q}) = \int_{\mathbb{S}_Q} t^* c_1(\mathcal{O}(D), \|\cdot\|) d\mu_{\text{Haar}}$ where t^* is the pull-back defined by the multiplication by t . It follows that if $c_1(\mathcal{O}(D), \|\cdot\|) \geq 0$ then $c_1(\mathcal{O}(D), \|\cdot\|_{\mathbb{S}_Q}) \geq 0$.

Let $\|\cdot\|$ be an admissible hermitian metric. By definition there exists $(\|\cdot\|_n)_{n \in \mathbb{N}}$ a sequence of smooth, semipositive hermitian metrics converging uniformly to $\|\cdot\|$. By averaging this sequence as before we get a sequence of smooth, semipositive and \mathbb{S}_Q -invariant hermitian metrics which converges uniformly to $\|\cdot\|$. \square

Let \overline{D} be a smooth positive toric divisor on X . We set $\Psi_D = -\log \|s_D\|_{\overline{D}}^2$. The exponential map gives the following change of variables, $z = \exp(-u - i\theta) \in \mathbb{T}_Q$ for any $z \in \mathbb{T}_Q$ where $u, \theta \in Q_{\mathbb{R}}$. Then $\frac{\partial^2 \Psi_D}{\partial z_k \partial \bar{z}_l} = \frac{1}{z_k \bar{z}_l} \frac{\partial^2 g_{\overline{D}}}{\partial u_k \partial u_l}$ for $k, l = 1, \dots, n$. Since \overline{D} is positive and smooth then $g_{\overline{D}}$ is a strictly concave smooth function on $Q_{\mathbb{R}}$. Hence, for any $x \in \text{Int}(\Delta_D)$ there exists a unique $G_{\overline{D}}(x) \in Q_{\mathbb{R}}$ such that $\check{g}(x) = x \cdot G_{\overline{D}}(x) - g_{\overline{D}}(G_{\overline{D}}(x))$, and we can show that $G_{\overline{D}}$ is smooth on Δ_D and $\frac{\partial g_{\overline{D}}}{\partial u} \circ G_{\overline{D}} = \text{Id}_{\Delta_D}$ (this follows from [9, Theorem 26.5]). In other words, $x := \frac{\partial g_{\overline{D}}}{\partial u}$ defines a \mathcal{C}^∞ -diffeomorphism between $\text{Int}(\Delta_D)$ and $Q_{\mathbb{R}}$, and we have

$$c_1(\overline{D})^{\wedge n} = \frac{n!}{(2\pi)^n} \det\left(\frac{\partial^2 g_{\overline{D}}}{\partial u_k \partial u_l}\right)_{1 \leq k, l \leq n} du_1 \wedge \dots \wedge du_n \wedge d\theta_1 \wedge \dots \wedge d\theta_n = \frac{n!}{(2\pi)^n} dx_1 \wedge \dots \wedge dx_n \wedge d\theta_1 \wedge \dots \wedge d\theta_n.$$

Let ϕ be a continuous function on X which is invariant under the action of \mathbb{S}_Q . We denote by ϕ_D the function on Δ_D given by $\phi_D(x) = \phi(\exp(-G_{\overline{D}}(x)))$. One can show that

$$\int_X \phi c_1(\overline{D})^{\wedge n} = n! \int_{\Delta_D} \phi_D dx,$$

where $dx = dx_1 \wedge \dots \wedge dx_n$ denotes the standard Lebesgue measure on $Q_{\mathbb{R}}$. In particular, one have the following identity $\deg_D(X) = n! \text{vol}(\Delta_D)$, which extends easily to nef divisors.

Lemma 3.2. *Let f be a smooth function on X and \mathbb{S}_Q -invariant. We have*

$$\int_X df \wedge d^c f \wedge c_1(\overline{D})^{n-1} = \int_{\Delta_D} \left\langle \frac{dG_{\overline{D}}(x)}{dx}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle dx,$$

where $f(x) := f(\exp(-G_{\overline{D}}(x)))$ on Δ_D .

Proof. Notice that $\frac{dG_{\overline{D}}(x)}{dx} = \text{Hess}(\check{g})$ and using the change of coordinates $x := \frac{dq_t}{dt}$ one can deduce the lemma. \square

Let $\|\cdot\|_{\overline{D}_0}$ and $\|\cdot\|_{\overline{D}_1}$ be two smooth, \mathbb{S}_Q -invariant and positive hermitian metrics on $\mathcal{O}(D)$ and we let $\overline{D}_i := (D, \|\cdot\|_{\overline{D}_i})$ for $i = 0, 1$. For any $t \in [0, 1]$ we set $g_t = tg_1 + (1-t)g_0$. Then we have the following result

Proposition 3.3. *The following function defined on $[0, 1]$*

$$t \mapsto \int_{\Delta_D} \check{g}_t dx,$$

is differentiable on $[0, 1[$. Moreover, we have

$$\frac{d}{dt} \left(\int_{\Delta_D} \check{g}_t dx \right) \Big|_{t=0^+} = - \int_{\Delta_D} (g_1 - g_0)(G_0(x)) dx.$$

Proof. We denote by \overline{D}_t the positive and smooth toric metrized divisor D endowed with g_t and we set $G_t := G_{\overline{D}_t}$ for any $t \in [0, 1]$. We have

$$\check{g}_t(x) = \langle x, G_t(x) \rangle - g_0(G_t(x)) - t(g_1 - g_0)(G_t(x)) \quad \forall x \in \Delta_D, \forall t \in [0, 1].$$

We set $\mathcal{F}_t(x) := \langle x, G_t(x) \rangle - g_0(G_t(x))$ for any $x \in \Delta_D$ and $t \in [0, 1]$. We have $\frac{d\mathcal{F}_t}{dt}(x) = \langle x, \frac{dG_t(x)}{dt} \rangle - \frac{dg_0}{du} \Big|_{u=G_t(x)} \cdot \frac{dG_t(x)}{dt} = \langle x - G_0^{-1}(G_t(x)), \frac{dG_t(x)}{dt} \rangle$. That is

$$\frac{d\mathcal{F}_t}{dt}(x) = \langle x - G_0^{-1}(G_t(x)), \frac{dG_t(x)}{dt} \rangle \quad \forall x \in \Delta_D, \forall t \in [0, 1]. \quad (2)$$

Let $u \in Q_{\mathbb{R}}$ such that $x = \frac{dg_t}{dt}(u)$. Then $x - G_0^{-1}(G_t(x)) = \frac{dg_t}{dt}(u) - \frac{dg_0}{dv}(u) = t(\frac{dg_1}{dt}(u) - \frac{dg_0}{dv}(u))$. Recall that $G_t(\frac{dg_t(v)}{dt}) = v$ for any $v \in Q_{\mathbb{R}}$. This gives $\frac{dG_t}{dt}(\frac{dg_t}{dv}) = -\frac{dG_t}{dx}(\frac{dg_t}{dv}) \cdot \frac{\partial(g_1 - g_0)}{\partial v}$. Then (2) becomes

$$\frac{d\mathcal{F}_t}{dt}(x) = -t \langle \frac{dG_t}{dx}(\frac{dg_t}{dv}(u)) \cdot (\frac{dg_1}{dv}(u) - \frac{dg_0}{dv}(u)), \frac{dg_1}{dv}(u) - \frac{dg_0}{dv}(u) \rangle.$$

Which is equivalent to

$$\frac{d\mathcal{F}_t}{dt}(x) = -t \langle \frac{dG_t}{dx}(x) \cdot (\frac{dg_1}{dv}(G_t(x)) - \frac{dg_0}{dv}(G_t(x))), \frac{dg_1}{dv}(G_t(x)) - \frac{dg_0}{dv}(G_t(x)) \rangle.$$

The function $f := \log \frac{\|\cdot\|_{\overline{D}_1}}{\|\cdot\|_{\overline{D}_0}}$ is smooth and \mathbb{S}_Q -invariant on X . Then by (3.2) we have

$$\langle \frac{dG_t}{dx}(x) \cdot (\frac{dg_1}{dv}(G_t(x)) - \frac{dg_0}{dv}(G_t(x))), \frac{dg_1}{dv}(G_t(x)) - \frac{dg_0}{dv}(G_t(x)) \rangle > dx \wedge d\theta = df \wedge d^c f \wedge c_1(\overline{D}_t)^{n-1}$$

which is absolutely integrable. Therefore,

$$\frac{d}{dt} \int_{\Delta_D} \mathcal{F}_t dx = \int_{\Delta_D} \frac{d\mathcal{F}_t}{dt} dx = -t \int_X d(\log \frac{\|\cdot\|_1}{\|\cdot\|_0}) \wedge d^c(\log \frac{\|\cdot\|_1}{\|\cdot\|_0}) \wedge c_1(\overline{D}_t)^{n-1} \quad \forall t \in [0, 1[.$$

With similar arguments we can establish that $\int_{\Delta_D} (g_1 - g_0)(G_t(x)) dx$ is also differentiable on $[0, 1[$. We conclude that

$$t \mapsto \int_{\Delta_D} \check{g}_t dx,$$

is differentiable, and we have

$$\frac{d}{dt} \left(\int_{\Delta_D} \check{g}_t dx \right) \Big|_{t=0^+} = - \int_{\Delta_D} (g_1 - g_0)(G_0(x)) dx.$$

□

Theorem 3.4. Let $\mathcal{O}(D)$ be an ample line bundle on X . Let $\|\cdot\|_{\overline{D}_1}$ and $\|\cdot\|_{\overline{D}_0}$ be two smooth, \mathbb{S}_Q -invariant and positive hermitian metrics on $\mathcal{O}(D)$. We have,

$$\mathcal{E}(\overline{D}_1) - \mathcal{E}(\overline{D}_0) = - \int_{\Delta_D} (\check{g}_{\overline{D}_1}(x) - \check{g}_{\overline{D}_0}(x)) dx.$$

Moreover, this equality extends to admissible metrics.

Proof. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two smooth, positive and \mathbb{S}_Q -invariant hermitian metrics on $\mathcal{O}(D)$. Let $s \in [0, 1]$ and let $\|\cdot\|_s$ be the metric defined by $g_s := (1-s)g_{\overline{D}_0} + sg_{\overline{D}_1}$. This metric is smooth, positive and \mathbb{S}_Q -invariant. We denote by \overline{D}_s the metrized toric divisor D endowed with the metric $\|\cdot\|_s$. Applying Proposition 3.3 one get for any $s \in [0, 1[$

$$\frac{d}{dt} \left(\int_{\Delta_D} \check{g}_t \right)_{|_{t=s+}} = - \int_{\Delta_D} (g_1 - g_0)(G_s(x)) dx.$$

From the definition of the Monge-Ampère functional, one get easily

$$\frac{d}{dt} (\mathcal{E}(\overline{D}_s) - \mathcal{E}(\overline{D}_0))_{|_{t=s+}} = \int_X (\log \frac{\|\cdot\|_0}{\|\cdot\|_1}) c_1(L, \|\cdot\|_s)^{\wedge n}.$$

Now, by using the change of variables $u = G_s(x)$. We get $\int_X (\log \frac{\|\cdot\|_0}{\|\cdot\|_1}) c_1(L, \|\cdot\|_s)^n = - \int_{\Delta_D} (g_1 - g_0)(G_s(x)) dx$. Thus,

$$\frac{d}{dt} (\mathcal{E}(\overline{D}_s) - \mathcal{E}(\overline{D}_0))_{|_{t=s+}} = \frac{d}{dt} \left(\int_{\Delta_D} \check{g}_t \right)_{|_{t=s+}} \quad \forall s \in [0, 1[.$$

Therefore

$$\mathcal{E}(\overline{D}_1) - \mathcal{E}(\overline{D}_0) = - \int_{\Delta_D} (\check{g}_{\overline{D}_1}(x) - \check{g}_{\overline{D}_0}(x)) dx. \quad (3)$$

Suppose now that $\|\cdot\|_0$ and $\|\cdot\|_1$ are admissible. By definition, there exists $(\|\cdot\|_{i,n})_{n \in \mathbb{N}}$ a sequence of smooth and semipositive metrics on L converging uniformly to $\|\cdot\|_i$, $i = 1, 2$. By Proposition 3.1, we can assume that the metrics of the sequences are \mathbb{S}_Q -invariant. Moreover, we can also suppose that $\|\cdot\|_{i,n}$ is positive for all $n \in \mathbb{N}$ and $i = 0, 1$. Indeed, let $\|\cdot\|'$ be a smooth, positive and \mathbb{S}_Q -invariant metric on $\mathcal{O}(D)$ (for example one can take the pull-back of the Fubini-Study by an equivariant morphism defined by D) then $\|\cdot\|_{i,n}^{1-1/n} \|\cdot\|'^{1/n}$ is positive, smooth and \mathbb{S}_Q -invariant. We have (3) holds for $g_i = g_{n,i}$ for any $n \in \mathbb{N}$ and $i = 0, 1$. By the theory of Bedford-Taylor the LHS converges to $\mathcal{E}(\overline{D}_1) - \mathcal{E}(\overline{D}_0)$. By (2.3) the RHS converges to $\int_{\Delta_D} (\check{g}_{\overline{D}_1}(x) - \check{g}_{\overline{D}_0}(x)) dx$. \square

Proposition 3.5. *Let \overline{D} be a toric metrized divisor and \overline{A} a toric metrized divisor such that A is effective. We have*

$$\lim_{l \in \mathbb{N}, l \rightarrow \infty} \frac{1}{l^{n+1}} \int_{l\Delta_D + \Delta_A} (lg_{\overline{D}} + g_{\overline{A}}) dx = \int_{\Delta_D} \check{g}_{\overline{D}} dx.$$

(dx denotes the standard Lebesgue measure on $Q_{\mathbb{R}}$).

Proof. We set $g_{l\overline{D} + \overline{A}} := lg_{\overline{D}} + g_{\overline{A}}$ and $g_{\overline{D} + \frac{1}{l}\overline{A}} := g_{\overline{D}} + \frac{1}{l}g_{\overline{A}}$ for any $l \in \mathbb{N}^*$.

The assumption that \overline{D} and \overline{A} are toric metrized divisors implies that $\sup_{v \in Q_{\mathbb{R}}} |g_{\overline{D}}(v) - g_{\overline{D}_{\infty}}(v)|$ and $\sup_{v \in Q_{\mathbb{R}}} |g_{\overline{A}}(v) - g_{\overline{A}_{\infty}}(v)|$ are finite. There exists a constant C such that $|\check{g}_{\overline{D} + \frac{1}{l}\overline{A}}| \leq C$ on $\Delta_D + \frac{1}{l}\Delta_A$ for any $l \in \mathbb{N}$. This follows from the following inequality

$$|\check{g}_{\overline{D} + \frac{1}{l}\overline{A}}| \leq \sup_{v \in Q_{\mathbb{R}}} |g_{\overline{D}}(v) - g_{\overline{D}_{\infty}}(v)| + \frac{1}{l} \sup_{v \in Q_{\mathbb{R}}} |g_{\overline{A}}(v) - g_{\overline{A}_{\infty}}(v)|,$$

on $\Delta_D + \frac{1}{l}\Delta_A$ which is a consequence of Proposition 2.3 combined with the fact that $\check{g}_{\overline{D}_{\infty} + \frac{1}{l}\overline{A}_{\infty}} = 0$ on $\Delta_D + \frac{1}{l}\Delta_A$. Notice that $0 \in \Delta_A$ because A is effective (this follows from [8, Proposition 2.1 (v)]). Then $\check{g}_{\overline{A}}(0)$ is finite and it follows that $g_{\overline{A}}$ is bounded from above. If we multiply the metric of \overline{A} by a positive constant, then it is possible to assume that $g_{\overline{A}} \leq 0$. Observe that the assertion of the proposition remains true.

By an obvious change of variables, we have

$$\frac{1}{l^{n+1}} \int_{l\Delta_D + \Delta_A} \check{g}_{l\overline{D} + \overline{A}} dx = \int_{\Delta_D + \frac{1}{l}\Delta_A} \check{g}_{\overline{D} + \frac{1}{l}\overline{A}} dx = \int_{\Delta_D} \check{g}_{\overline{D} + \frac{1}{l}\overline{A}} dx + \int_{(\Delta_D + \frac{1}{l}\Delta_A) \setminus \Delta_D} \check{g}_{\overline{D} + \frac{1}{l}\overline{A}} dx.$$

Fix $x \in \Delta_D$. Since $0 \in \Delta_A$ and $g_{\overline{A}} \leq 0$, then $\check{g}_{\overline{D} + \frac{1}{l}\overline{A}}(x) \geq \check{g}_{\overline{D} + \frac{1}{l'}\overline{A}}(x) \geq \check{g}_{\overline{D}}(x)$ for any $l \leq l'$ in \mathbb{N}^* . Let $u \in Q_{\mathbb{R}}$ such that $\check{g}_{l\overline{D}}(x) = x \cdot u - g_{\overline{D}}(u)$. Then

$$\check{g}_{\overline{D} + \frac{1}{l}\overline{A}}(x) \leq \check{g}_{\overline{D}}(x) - \frac{1}{l} g_{\overline{A}}(u) \quad \forall l \in \mathbb{N}^*.$$

Therefore $\check{g}_{\overline{D} + \frac{1}{l}\overline{A}}$ is a decreasing function converging pointwise to $\check{g}_{\overline{D}}$ on Δ_D . Since $0 \in \Delta_A$ and $|\check{g}_{\overline{D} + \frac{1}{l}\overline{A}}| \leq C$ on Δ_D we conclude (by using the Fatou-Lebesgue theorem) that

$$\lim_{l \rightarrow \infty} \int_{\Delta_D} \check{g}_{\overline{D} + \frac{1}{l}\overline{A}} = \int_{\Delta_D} \check{g}_{\overline{D}}.$$

On other hand, we have

$$\left| \int_{(\Delta_D + \frac{1}{l}\Delta_A) \setminus \Delta_D} \check{g}_{\overline{D} + \frac{1}{l}\overline{A}} \right| \leq C \text{vol}((\Delta_D + \frac{1}{l}\Delta_A) \setminus \Delta_D).$$

Therefore

$$\lim_{l \rightarrow \infty} \int_{\Delta_D + \frac{1}{l}\Delta_A} (g_{\overline{D}} + \frac{1}{l} g_{\overline{A}})^\vee = \int_{\Delta_D} \check{g}_{\overline{D}}.$$

□

Corollary 3.6. *Let $\mathcal{O}(D)$ be a big and nef line bundle on X . Let $\|\cdot\|_{\overline{D}_1}$ and $\|\cdot\|_{\overline{D}_0}$ be two admissible and \mathbb{S}_Q -invariant hermitian metrics on $\mathcal{O}(D)$ and set $\overline{D}_i = (D, \|\cdot\|_{\overline{D}_i})$ for $i = 0, 1$. We have,*

$$\mathcal{E}(\overline{D}_1) - \mathcal{E}(\overline{D}_0) = - \int_{\Delta_D} (\check{g}_{\overline{D}_1}(x) - \check{g}_{\overline{D}_0}(x)) dx.$$

Proof. Let \overline{A} be a positive and smooth toric metrized divisor. We have $\overline{A} + l\overline{D}_i$ is a positive continuous toric metrized divisor, for $i = 0, 1$ and any $l \in \mathbb{N}$. Then by Theorem 3.4,

$$\mathcal{E}(\overline{A} + l\overline{D}_1) - \mathcal{E}(\overline{A} + l\overline{D}_0) = - \int_{\Delta_{A+lD}} (\check{g}_{\overline{A}+l\overline{D}_1}(x) - \check{g}_{\overline{A}+l\overline{D}_0}(x)) dx \quad \forall l \in \mathbb{N}.$$

We have $\mathcal{E}(\overline{A} + l\overline{D}_1) - \mathcal{E}(\overline{A} + l\overline{D}_0) = l^{n+1}(\mathcal{E}(\overline{D}_1) - \mathcal{E}(\overline{D}_0)) + O(l^n), \forall l \in \mathbb{N}$. Since A is effective and by Proposition 3.5 we conclude that

$$\mathcal{E}(\overline{D}_1) - \mathcal{E}(\overline{D}_0) = - \int_{\Delta_D} (\check{g}_{\overline{D}_1}(x) - \check{g}_{\overline{D}_0}(x)) dx.$$

□

Lemma 3.7. *Let Θ be a convex compact subset in \mathbb{R}^n such that $\text{vol}(\Theta) > 0$ (vol denotes the volume induced by the standard Lebesgue measure dx of \mathbb{R}^n) and let φ be a bounded concave function on Θ . We have,*

$$\lim_{l \in \mathbb{N}^*, l \rightarrow \infty} \frac{1}{l^n} \sum_{e \in l\Theta \cap P} \varphi\left(\frac{e}{l}\right) = \int_{\Theta} \varphi.$$

Proof. Let $\varepsilon > 0$. There exists Θ' a convex compact subset in $\text{Int}(\Theta)$ such that $\text{vol}(\Theta \setminus \Theta') < \varepsilon$. By [9, Theorem 10.1], the concave function φ is continuous on Θ' . Then $|\lim_{l \in \mathbb{N}^*, l \rightarrow \infty} \frac{1}{l^n} \sum_{\mathbf{e} \in l\Theta' \cap P} \varphi(\frac{\mathbf{e}}{l}) - \int_{\Theta'} \varphi| \leq \varepsilon$ for $l \gg 1$. We have

$$\begin{aligned} \left| \frac{1}{l^n} \sum_{\mathbf{e} \in l\Theta \cap P} \varphi(\frac{\mathbf{e}}{l}) - \int_{\Theta} \varphi \right| &\leq \left| \frac{1}{l^n} \sum_{\mathbf{e} \in l\Theta' \cap P} \varphi(\frac{\mathbf{e}}{l}) - \int_{\Theta'} \varphi \right| + \left| \int_{\Theta} \varphi - \int_{\Theta'} \varphi \right| + \left| \frac{1}{l^n} \sum_{\mathbf{e} \in l(\Theta \setminus \Theta') \cap P} \varphi(\frac{\mathbf{e}}{l}) \right| \\ &\leq \left| \frac{1}{l^n} \sum_{\mathbf{e} \in l\Theta' \cap P} \varphi(\frac{\mathbf{e}}{l}) - \int_{\Theta'} \varphi \right| + \sup_{\Theta} |\varphi| \text{vol}(\Theta \setminus \Theta') + \sup_{\Theta} |\varphi| \left(\frac{1}{l^n} \#(l\Theta \cap P) - \frac{1}{l^n} \#(l\Theta' \cap P) \right), \end{aligned}$$

and since $\lim_{l \rightarrow \infty} \frac{1}{l^n} \#(l\Theta \cap P) = \text{vol}(\Theta)$ and $\lim_{l \rightarrow \infty} \frac{1}{l^n} \#(l\Theta' \cap P) = \text{vol}(\Theta')$ we conclude that

$$\frac{1}{l^n} \sum_{\mathbf{e} \in l\Theta \cap P} \varphi(\frac{\mathbf{e}}{l}) - \int_{\Theta} \varphi = O(\varepsilon) \quad \forall l \gg 1.$$

□

Lemma 3.8. *Let Θ be a convex compact subset in \mathbb{R}^n such that $\text{vol}(\Theta) > 0$. For any $l \in \mathbb{N}^*$, let $A_l = (a_{\mathbf{e}, \mathbf{e}'})_{\mathbf{e}, \mathbf{e}' \in l\Theta \cap P}$ be a positive symmetric matrix indexed by $l\Theta \cap P$, and let K_l be a subset of $\mathbb{R}^{l\Theta \cap P} \simeq \mathbb{R}^{\#(l\Theta \cap P)}$ given by*

$$K_l = \{ (x_{\mathbf{e}}) \in \mathbb{R}^{l\Theta \cap P} \mid \sum_{\mathbf{e}, \mathbf{e}' \in l\Theta \cap P} a_{\mathbf{e}, \mathbf{e}'} x_{\mathbf{e}} x_{\mathbf{e}'} \leq 1 \}.$$

We assume there is an integrable function $\varphi : \Theta \rightarrow \mathbb{R}$ such that for any $\varepsilon > 0$, there exists a constant D verifying

$$|\log(\frac{1}{a_{\mathbf{e}, \mathbf{e}}}) - l\varphi(\frac{\mathbf{e}}{l})| \leq D + \varepsilon l,$$

for any $l \gg 1$ and $\mathbf{e} \in l\Theta \cap P$. Then, we have

$$\lim_{l \rightarrow \infty} \frac{1}{l^{n+1}} \sum_{\mathbf{e} \in l\Theta \cap P} \log(\frac{1}{a_{\mathbf{e}, \mathbf{e}}}) = \int_{\Theta} \varphi(x) dx.$$

Proof. Let $\varepsilon > 0$. By assumption, there exists constant D such that

$$\varphi(\frac{\mathbf{e}}{l}) - \frac{1}{l} D - \varepsilon \leq \frac{1}{l} \log(\frac{1}{a_{\mathbf{e}, \mathbf{e}}}) \leq \varphi(\frac{\mathbf{e}}{l}) + \frac{1}{l} D + \varepsilon \quad \forall l \gg 1 \quad \forall \mathbf{e} \in l\Theta \cap P.$$

Then

$$\frac{1}{l^d} \sum_{\mathbf{e} \in l\Theta \cap P} \varphi(\frac{\mathbf{e}}{l}) - \frac{m_l}{l^{n+1}} D - \frac{m_l}{l^n} \varepsilon \leq \frac{1}{l^{n+1}} \sum_{\mathbf{e} \in l\Theta \cap P} \log(\frac{1}{a_{\mathbf{e}, \mathbf{e}}}) \leq \frac{1}{l^n} \sum_{\mathbf{e} \in l\Theta \cap P} \varphi(\frac{\mathbf{e}}{l}) + \frac{m_l}{l^{n+1}} D + \frac{m_l}{l^n} \varepsilon \quad \forall l \gg 1.$$

where $m_l = \#(l\Theta \cap P)$. Note that

$$\lim_{l \rightarrow \infty} \frac{1}{l^n} \sum_{\mathbf{e} \in l\Theta \cap P} \varphi(\frac{\mathbf{e}}{l}) = \lim_{l \rightarrow \infty} \sum_{x \in \Theta \cap (1/l)P} \varphi(x) = \int_{\Theta} \varphi(x) dx. \quad (4)$$

Then, we can find $l_0 \gg 1$ such that

$$\left| \frac{1}{l^{n+1}} \sum_{\mathbf{e} \in l\Theta \cap P} \log(\frac{1}{a_{\mathbf{e}, \mathbf{e}}}) - \int_{\Theta} \varphi(x) dx \right| \leq \varepsilon + \frac{m_l}{l^{n+1}} D + \frac{m_l}{l^n} \varepsilon \quad \forall l \geq l_0.$$

Since $m_l = O(l^n)$. We can deduce that,

$$\left| \frac{1}{l^{n+1}} \sum_{\mathbf{e} \in l\Theta \cap P} \log\left(\frac{1}{a_{\mathbf{e}, \mathbf{e}}}\right) - \int_{\Theta} \varphi(x) dx \right| = O(\varepsilon) \quad \forall l \geq l_0.$$

Then

$$\lim_{l \rightarrow \infty} \frac{1}{l^{n+1}} \sum_{\mathbf{e} \in l\Theta \cap P} \log\left(\frac{1}{a_{\mathbf{e}, \mathbf{e}}}\right) = \int_{\Theta} \varphi(x) dx.$$

□

4 THE PROOF OF THE MAIN THEOREM

Let X be a complex projective nonsingular toric variety and D a toric divisor on X such that $\mathcal{O}(D)$ is big and nef. Let $\|\cdot\|_{\overline{D}_0}$ and $\|\cdot\|_{\overline{D}_1}$ be two continuous toric metrics on D and $\overline{D}_i := (D, \|\cdot\|_{\overline{D}_i})$ and $\phi_{\overline{D}_i}$ the associated weight for $i = 0, 1$. Let μ_j be a probability measure \mathbb{S}_Q -invariant on X with the Bernstein-Markov property with respect to $\|\cdot\|_{\overline{D}_j}$, $j = 0, 1$. We denote by \overline{D}_{iX} the metrized toric divisor endowed with the weight $P_X \phi_{\overline{D}_i}$ for $i = 0, 1$.

Proposition 4.1. *We have,*

$$\lim_{k \rightarrow \infty} \mathcal{L}_k(\mu_1, \overline{D}_1) - \mathcal{L}_k(\mu_0, \overline{D}_0) = -\frac{1}{\text{vol}(D)} \int_{\Delta_D} (\check{g}_{\overline{D}_1} - \check{g}_{\overline{D}_0}) dx. \quad (5)$$

Proof. For any $\mathbf{e} \in \Delta_D \cap P$, we set $s_{\mathbf{e}} := \|\chi^{\mathbf{e}}\|_{L^2(\mu_1, \overline{D}_1)}^{-1} \chi^{\mathbf{e}}$ ($\chi^{\mathbf{e}}$ is the global section of $\mathcal{O}(D)$ defined by \mathbf{e}). Recall that $H^0(X, \mathcal{O}(D)) = \oplus_{m \in \Delta_D \cap P} \mathbb{C} \chi^m$. Since μ_0 and μ_1 are \mathbb{S}_Q -invariant, then $\{s_{\mathbf{e}} | \mathbf{e} \in \Delta_D \cap P\}$ is an $L^2(\mu_1, \overline{D}_1)$ -orthonormal basis of $H^0(X, L)$ and we have

$$\frac{\text{vol}_{L^2(\mu_1, \overline{D}_1)}(B(\mu_1, \overline{D}_1))}{\text{vol}_{L^2(\mu_0, \overline{D}_0)}(B(\mu_0, \overline{D}_0))} = \det((s_{\mathbf{e}}, s_{\mathbf{e}'}))_{L^2(\mu_0, \overline{D}_0)} = \prod_{\mathbf{e} \in \Delta_D \cap P} (s_{\mathbf{e}}, s_{\mathbf{e}})_{L^2(\mu_0, \overline{D}_0)}.$$

Then for any $k \in \mathbb{N}^*$,

$$\mathcal{L}_k(\mu_1, \overline{D}_1) - \mathcal{L}_k(\mu_0, \overline{D}_0) = \frac{1}{k N_k} \log \prod_{\mathbf{e} \in k\Delta_D \cap P} (s_{\mathbf{e}}, s_{\mathbf{e}})_{L^2(\mu_0, \overline{D}_0)},$$

($N_k := \dim H^0(X, \mathcal{O}(kD))$). Since (μ_0, \overline{D}_0) and (μ_1, \overline{D}_1) satisfy the Bernstein-Markov property, then for any $\varepsilon > 0$ there exists a constant D such that

$$|\log(s_{\mathbf{e}}, s_{\mathbf{e}})_{L^2(\mu_0, \overline{D}_0)} - k(\check{g}_{\overline{D}_1} - \check{g}_{\overline{D}_0})\left(\frac{\mathbf{e}}{k}\right)| \leq D + k\varepsilon, \quad \forall \mathbf{e} \in k\Delta_D \cap P, \forall k \gg 1.$$

(Notice that we use the fact $\|s_{\mathbf{e}}\|_{\text{sup}, k\overline{D}} = \exp(-k\check{g}_{\overline{D}}(\frac{\mathbf{e}}{k}))$, see the proof of lemma 2.4). Now let $\Theta := \Delta_D$ and $\phi := \check{g}_{\overline{D}_1} - \check{g}_{\overline{D}_0}$. They satisfy the assumptions of lemma 3.8 (More precisely, ϕ satisfies (4) which is a consequence of lemma 3.7). This concludes the proof of the proposition. □

4.1 THE CASE OF AMPLE DIVISOR

Suppose that D is ample. We know that $\|\cdot\|_{\overline{D}_i X}$ is a continuous and semipositive metric on D for $i = 0, 1$. By [7, Theorem 4.6.1], this metric is admissible. Using Corollary 3.6, we deduce that

$$\mathcal{E}(\overline{D}_{1X}) - \mathcal{E}(\overline{D}_{0X}) = - \int_{\Delta_D} (\check{g}_{\overline{D}_{1X}}(x) - \check{g}_{\overline{D}_{0X}}(x)) dx.$$

Thus,

$$\mathcal{E}(\overline{D}_{1X}) - \mathcal{E}(\overline{D}_{0X}) = - \int_{\Delta_D} (\check{g}_{\overline{D}_1}(x) - \check{g}_{\overline{D}_0}(x)) dx,$$

by lemma 2.4. Now by Proposition 4.1 we get

$$\lim_{k \rightarrow \infty} \mathcal{L}_k(\mu_1, \overline{D}_1) - \mathcal{L}_k(\mu_0, \overline{D}_0) = \frac{1}{\text{vol}(D)} (\mathcal{E}(\overline{D}_{1X}) - \mathcal{E}(\overline{D}_{0X})) = \mathcal{E}_{\text{eq}}(\overline{D}_1) - \mathcal{E}_{\text{eq}}(\overline{D}_0).$$

Thus we proved Theorem 1.2 for ample divisors.

4.2 THE CASE OF BIG AND NEF DIVISOR

Let \overline{D} a metrized toric divisor such that D is big and nef. Let \overline{A} be a positive metrized toric divisor on X . Let $l > 0$ and we set $\phi_l := P_X(\phi_{\overline{D}} + \frac{1}{l}\phi_A) - \frac{1}{l}\phi_A$ where $\phi_{\overline{D}}$ (resp. ϕ_A) denotes the associated weight to \overline{D} (resp. \overline{A}). We have ϕ_l defines a continuous metric on D . Indeed, this is follows from the fact that $lD + A$ is ample for any $l \in \mathbb{N}$ (because D is nef and A is ample), which implies that $P_X(l\phi_{\overline{D}} + \phi_A)$ is a continuous weight on $lD + A$. We denote by \overline{D}'_l the continuous metrized toric divisor D endowed with the continuous weight ϕ_l .

Lemma 4.2. $(\phi_l)_{l \in \mathbb{N}^*}$ is a decreasing sequence of continuous weights on $\mathcal{O}(D)$ converging pointwise to $P_X\phi$.

Proof. First notice that the limit of the sequence $(\phi_l)_{l \in \mathbb{N}^*}$, if it exists, doesn't depend on the choice of the metric on A . Indeed, let $\phi_{1,A}$ and $\phi_{0,A}$ be two weights on A defining continuous metrics and we set $\phi_{i,l} := P_X(\phi_{\overline{D}} + \frac{1}{l}\phi_{i,A}) - \frac{1}{l}\phi_{i,A}$ for $i = 0, 1$. Then by Proposition 2.1 we have

$$|\phi_{1,l} - \phi_{0,l}| \leq \frac{2}{l} \sup_{x \in X} |\phi_{1,A} - \phi_{0,A}|, \quad \forall l \in \mathbb{N}^*.$$

Suppose that ϕ_A is psh. Let ψ be a psh weight on D with $\psi \leq \phi_{\overline{D}}$ then $\psi + \frac{1}{l}\phi_A$ is also a psh weight satisfying $\psi + \frac{1}{l}\phi_A \leq P_X(\phi_A + \frac{1}{l}\phi_A)$. Therefore, $\psi \leq P_X(\phi_{\overline{D}} + \frac{1}{l}\phi_A) - \frac{1}{l}\phi_A$ for any ψ a psh weight on D such that $\psi \leq \phi_{\overline{D}}$. Thus

$$P_X\phi_{\overline{D}} \leq \phi_l \leq \phi_{\overline{D}} \quad \forall l \in \mathbb{N}^*.$$

Let $l \geq k$, then clearly $P_X(\phi_{\overline{D}} + \frac{1}{l}\phi_A) + (\frac{1}{k} - \frac{1}{l})\phi_A \leq P_X(\phi_{\overline{D}} + \frac{1}{k}\phi_A)$. So

$$\phi_l \leq \phi_k \quad \forall l \geq k.$$

If we set $\Psi := \lim_{l \in \mathbb{N}^*, l \rightarrow \infty} \phi_l$, then

$$P_X\phi \leq \Psi \leq \phi_{l+1} \leq \phi_l \leq \phi_{\overline{D}} \quad \forall l \in \mathbb{N}^*. \quad (6)$$

Let $k \in \mathbb{N}^*$, we have $\phi_l + \frac{1}{k}\phi_A$ is psh for any $l \geq k$. Then $\Psi + \frac{1}{k}\phi_A$ is also a psh function for any $k \in \mathbb{N}^*$ (see for instance [5, Theorem 5.4]). It follows that Ψ is psh weight on D . We conclude that

$$P_X\phi_{\overline{D}} = \Psi.$$

□

Recall that ϕ_l is continuous for any $l \in \mathbb{N}^*$. We have $(\phi_l + \frac{1}{k}\phi_A)_{l \geq k}$ is a decreasing sequence of continuous psh functions converging pointwise to $P_X\phi_{\overline{D}} + \frac{1}{k}\phi_A$ with minimal singularities. Then as l tends to ∞

$$\mathcal{E}(\overline{D}'_l + \frac{1}{k}\overline{A}) \rightarrow \mathcal{E}(\overline{D}_X + \frac{1}{k}\overline{A}), \quad (7)$$

for any $k \in \mathbb{N}^*$. The proof of the last assertion is similar to [2, Proposition 4.3] which is a consequence of the continuity of the Monge-Ampère operator, see [4].

Let $\phi_{0,D}$ be a continuous and \mathbb{S}_Q -invariant psh weight on D and we set \overline{D}' the metrized toric $(D, \phi_{0,D})$. We set $g_{k,l} := g_{\overline{D}'_l + \frac{1}{k}\overline{A}}$ for any $k, l \in \mathbb{N}^*$. Then by (2.4) and (6) we get

$$\check{g}_{\overline{D}_X + \frac{1}{k}\overline{A}} \leq \check{g}_{k,l+1} \leq \check{g}_{k,l} \leq \check{g}_{\overline{D} + \frac{1}{k}\overline{A}}. \quad (8)$$

By Theorem 3.4 we have

$$\mathcal{E}(\overline{D}'_l + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\phi_A) = - \int_{\Delta_D + \frac{1}{k}\Delta_A} (\check{g}_{k,l}(x) - \check{g}_{\overline{D}_0 + \frac{1}{k}\overline{A}}(x))dx.$$

Now using (8), get for any $l, k \in \mathbb{N}^*$,

$$- \int_{\Delta_D + \frac{1}{k}\Delta_A} (\check{g}_{\overline{D} + \frac{1}{k}\overline{A}} - \check{g}_{\overline{D}_0 + \frac{1}{k}\overline{A}})dx \leq \mathcal{E}(\overline{D}_l + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\overline{A}) \leq - \int_{\Delta_D + \frac{1}{k}\Delta_A} (\check{g}_{\overline{D}_X + \frac{1}{k}\overline{A}} - \check{g}_{\overline{D}_0 + \frac{1}{k}\overline{A}})dx.$$

As l tends to ∞ , we have by (7)

$$- \int_{\Delta_D + \frac{1}{k}\Delta_A} (\check{g}_{\overline{D} + \frac{1}{k}\overline{A}} - \check{g}_{\overline{D}_0 + \frac{1}{k}\overline{A}}(x))dx \leq \mathcal{E}(\overline{D}_X + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\overline{A}) \leq - \int_{\Delta_D + \frac{1}{k}\Delta_A} (\check{g}_{\overline{D}_X + \frac{1}{k}\overline{A}}(x) - \check{g}_{\overline{D}_0 + \frac{1}{k}\overline{A}}(x))dx,$$

for any $k \in \mathbb{N}^*$. By Proposition 3.5, we get

$$\begin{aligned} - \int_{\Delta_D} (\check{g}_{\overline{D}} - \check{g}_{\overline{D}_0})dx &\leq \liminf_k \mathcal{E}(\overline{D}_X + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\overline{A}) \\ &\leq \limsup_k \mathcal{E}(\overline{D}_X + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\overline{A}) \leq - \int_{\Delta_D} (\check{g}_{\overline{D}_X} - \check{g}_{\overline{D}_0})dx. \end{aligned}$$

Using lemma 2.4, we deduce that

$$\lim_k \mathcal{E}(\overline{D}_X + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\overline{A}) = - \int_{\Delta_D} (\check{g}_{\overline{D}} - \check{g}_{\overline{D}_0})dx. \quad (9)$$

Since $P_X\phi_{\overline{D}} - \phi_{\infty,D}$ is bounded then, by lemma 4.3 below, there exists a constant C such that

$$|\mathcal{E}(\overline{D}_X + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}_X) + \mathcal{E}(\overline{D}')| \leq \frac{C}{k} \quad \forall k \in \mathbb{N}^*.$$

Then

$$\liminf_{l \rightarrow \infty} \mathcal{E}(\overline{D}_X + \frac{1}{k}\overline{A}) - \mathcal{E}(\overline{D}' + \frac{1}{k}\overline{A}) = \mathcal{E}(\overline{D}_X) - \mathcal{E}(\overline{D}').$$

Combined with (9), we obtain

$$- \int_{\Delta_D} (\check{g}_{\overline{D}_X}(x) - \check{g}_{\overline{D}_0}(x))dx = \mathcal{E}(\overline{D}_X) - \mathcal{E}(\overline{D}_0).$$

We conclude that,

$$\lim_{k \rightarrow \infty} \mathcal{L}_k(\mu_1, \overline{D}_1) - \mathcal{L}_k(\mu_0, \overline{D}_0) = \mathcal{E}_{\text{eq}}(\overline{D}_1) - \mathcal{E}_{\text{eq}}(\overline{D}_0).$$

This ends the proof of Theorem 1.2.

Lemma 4.3. *Let D be a big and nef divisor and A an ample divisor on X . Let ψ_D and $\phi_{0,D}$ be two psh weight with minimal singularities on D . Let ϕ_A and $\phi_{0,A}$ be two positive continuous weight on A . We assume that $\phi_{0,D}, \phi_A$ and $\phi_{0,A}$ are continuous and $\psi_D - \phi_{\infty,D}$ is bounded. Then there exists a real polynomial P of degree $\leq n$ depending only on $\phi_A, \psi_D, \phi_{0,D}$ such that*

$$|\mathcal{E}(k\overline{D}_\psi + \overline{A}) - \mathcal{E}(k\overline{D}' + \overline{A}) - k^{n+1}(\mathcal{E}(\overline{D}_\psi) - \mathcal{E}(\overline{D}'))| \leq P(k) \quad \forall k \in \mathbb{N},$$

where $\overline{D}_\psi, \overline{D}'$ and \overline{A} are the metrized toric divisors endowed with $\psi_D, \phi_{0,D}$ and ϕ_A respectively.

Proof. We have

$$\begin{aligned} \mathcal{E}(k\overline{D}_\psi + \overline{A}) - \mathcal{E}(k\overline{D}' + \overline{A}) &= \sum_{i=0}^n \int_X (k(\psi_D - \phi_{0,D})) dd^c(k\psi_D + \phi_A)^i dd^c(k\phi_{0,D} + \phi_A)^{n-i} \\ &= k^{n+1}(\mathcal{E}(\overline{D}_\psi) - \mathcal{E}(\overline{D}')) + \int_X (\psi_D - \phi_{0,D})T, \end{aligned}$$

where T is a linear sum of terms of the form $dd^c(\phi_A)^\alpha dd^c(\phi_{0,D})^\beta dd^c(\psi_D)^\gamma k^\varepsilon$ with $\alpha, \beta, \gamma, \varepsilon \in \mathbb{N}$ and $\alpha + \beta + \gamma = n$ and $\varepsilon \leq n$.

We set $f_D := \psi_D - \phi_{0,D}$. This function is bounded since we assume that $\psi_D - \phi_{\infty,D}$ is bounded and $\phi_{0,D}$ is continuous. Then,

$$\left| \int_X f_D dd^c(\phi_A)^\alpha dd^c(\phi_{0,D})^\beta dd^c(\psi_D)^\gamma \right| \leq \sup_X |f_D| \int_X c_1(A)^\alpha c_1(D)^{\beta+\gamma}.$$

The lemma follows from the last inequality. □

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